Basic principles of celestial navigation

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Celestial navigation is a technique for determining one's geographic position by the observation of identified stars, identified planets, the Sun, and the Moon. This subject has a multitude of refinements which, although valuable to a professional navigator, tend to obscure the basic principles. I describe these principles, give an analytical solution of the classical two-star-sight problem without any dependence on prior knowledge of position, and include several examples. Some approximations and simplifications are made in the interest of clarity.

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I. INTRODUCTION

Celestial navigation is a technique for determining one's geographic position by the observation of identified stars, identified planets, the Sun, and the Moon. Its basic principles are a combination of rudimentary astronomical knowledge and spherical trigonometry.[1–3]

Anyone who has been on a ship that is remote from any terrestrial landmarks needs no persuasion on the value of celestial navigation. There are modern electronic navigational aids such as Loran, ocean bottom soundings, and the global positioning system (GPS) which depends on receiving precise timing data from a world-wide network of artificial satellites, each carrying one or more atomic clocks. This paper illustrates what can be learned with only a sextant, rudimentary astronomical tables, and an accurate knowledge of absolute time (from radio signals or a precise chronometer). It is intended for students who are curious about the basic principles of celestial navigation and makes no pretense of serving a professional navigator or surveyor.

II. THE EARTH AND GEOGRAPHIC POSITION

The surfaces of the land masses of the Earth and of the oceans are approximated as the surface of a spherically symmetric body of unit radius, hereafter called the terrestrial sphere.[4] The Earth as a whole is assumed to be rotating at a constant angular velocity about an inertially oriented axis through its geometric center O. A geographic position P is referenced to a right-hand spherical coordinate system whose center is at O and whose positive Z-axis is coincident with the Earth's rotational axis and in the same sense as the Earth's angular momentum vector. The plane through O that is perpendicular to this axis is called the equatorial plane (the $X-Y$ plane). Any plane that contains the rotational axis intersects the spherical surface in a great circle called a meridian. By convention, the X axis lies in the meridian plane (the prime meridian) that passes through a point in Greenwich, England.[1]

The position P is specified by two coordinates, its latitude Φ and longitude Λ. The latitude is the plane angle whose apex is at O, with one radial line passing through P and the other through the point at which the observer's meridian intersects the equator. Northern latitudes are taken to be positive (0° to 90°) and southern latitudes negative (0° to $-90°$). The longitude is the dihedral angle,[3,5–7] measured eastward from the prime meridian to the meridian through P. The longitude Λ is between 0° and 360°, although often it is convenient to take the longitude westward of the prime meridian to be between 0° and $-180°$. The longitude of P also can be specified by the plane angle in the equatorial plane whose vertex is at O with one radial line through the point at which the meridian through P intersects the equatorial plane and the other radial line through the point G at which the prime meridian intersects the equatorial plane (see Fig. 1).

III. THE CELESTIAL SPHERE

The celestial sphere is an imaginary spherical surface whose radius is very much greater than that of the Earth, with its center at O and its polar axis coincident with the Earth's rotational axis. The position of each celestial object is represented by the point on the celestial sphere at which a line to it from O intersects this sphere. Inasmuch as the distances to all stars, planets, and the Sun are much greater than the radius of the Earth, a terrestrial observer may be thought of as viewing the sky from O. For the nearby Moon, however, its apparent position on the celestial sphere is, because of parallax, different by as much as nearly one degree for observers at different latitudes and longitudes.[8] It is impossible to make an *a priori* correction for this effect if the observer's position is initially unknown.

On the celestial sphere, the declination δ of a celestial object is strictly analogous to the latitude Φ of a terrestrial observer as defined in the above. The second spherical coordinate of a celestial object, called right ascension α, is the dihedral angle analogous to eastward terrestrial longitude, but with the fundamental difference that it is measured from a different reference point. That reference point on the celestial equator is called the vernal equinox, usually denoted by γ, and defined as follows. The equatorial plane of the Earth is tilted to the plane of its orbit (the ecliptic plane) about the Sun by 23.44°. These two planes intersect along a line that pierces the celestial sphere at two points, called nodes. The vernal equinox is the ascending node of the ecliptic on the equator, that is, the position of the Sun on or about 21 March of each year as the Sun ascends from southerly to northerly declination. The right ascension α of a celestial object, always taken to be positive eastward, is measured from the vernal equinox. Astronomers usually measure right ascension in hours, minutes, and seconds of time, but it can be converted to the equivalent angular measure by taking one hour $=15°$. Another commonly used quantity is called the hour angle. It is the dihedral angle between any specified meridian

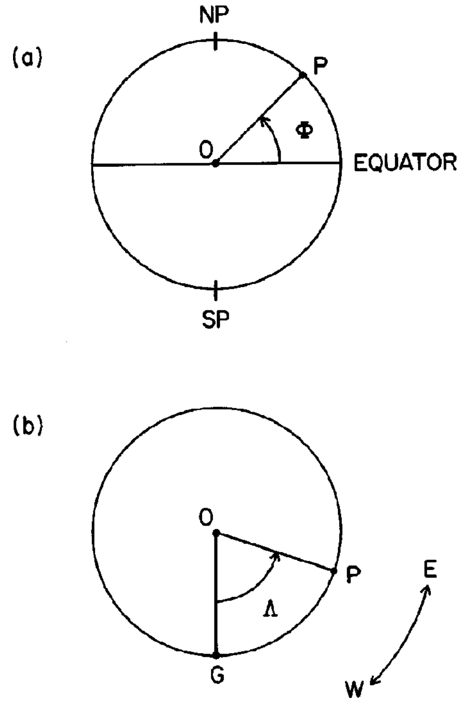

Fig. 1. Definition of latitude Φ and longitude Λ. (a) Meridian cross-section of the Earth through the observer at *P*. The center of the Earth is at *O* and NP and SP are the north pole and south pole, respectively. (b) Equatorial cross-section of the Earth. *G* represents the point at which the prime meridian through Greenwich intersects the equator and *P* represents the point at which the meridian through *P* intersects the equator. *E* means eastward and *W* means westward. The longitude Λ is the dihedral angle (Ref. 7) between the two named meridian planes, but is more easily visualized in this equatorial diagram.

plane and the meridian plane through a celestial object, always taken to be positive westward and lying in the range 0–24 h or 0°–360°. The sidereal hour angle (SHA) of a celestial object, taken to be positive westward from the vernal equinox, is related to the right ascension (Fig. 2) by

$$\mathrm{SHA} = 360° - \alpha. \tag{1}$$

We will take α, δ, and the SHA to be fixed quantities for any object (for example, stars or galaxies) beyond the solar system; however, they vary with time for the planets, the Sun, and the Moon. Their values are tabulated in almanacs,[9–11] of which *The Air Almanac*[9] is the most convenient for a navigator.

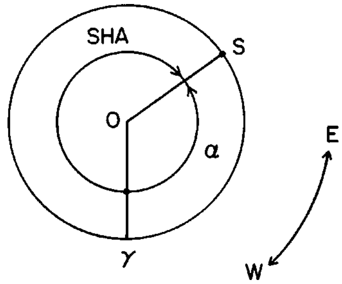

Fig. 2. Right ascension (α) and sidereal hour angle (SHA) are both dihedral angles (Ref. 7) measured from the meridian plane through the vernal equinox γ, with α positive eastward and SHA positive westward. In this equatorial diagram, *S* represents the intersection of the meridian plane through a star or any other celestial object with the equator.

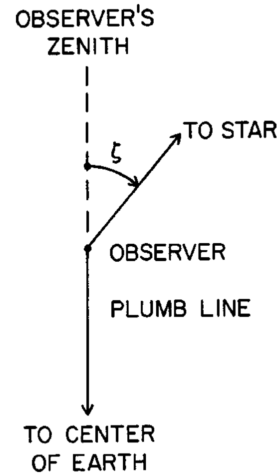

Fig. 3. Definition of the zenith distance ζ of a star or other celestial object.

IV. OBSERVATIONAL CONSIDERATIONS

Imagine that a terrestrial observer is located at a fixed point *P* of unknown latitude Φ and longitude Λ. The celestial sphere rotates westward from the observer's point of view at an angular rate such that the vernal equinox transits (passes through) the observer's meridian from east to west at intervals of 23 hour, 56 minute, 4 second of mean solar time, also called the sidereal rotational period of the Earth or the sidereal day.

The observer's zenith is defined by the outward extension of the line from the center of the Earth through the observer as in Fig. 3. The instantaneous angle between the observer's zenith and the line to a star is called the star's zenith distance and is denoted by ζ. As the celestial sphere rotates westward about the Earth (that is, clockwise as viewed from above the north celestial pole), ζ decreases from 90° as the star rises from below the observer's eastern horizon, has a minimum value $\zeta_{\min}$ as *S* crosses the observer's meridian, then increases to 90° as the star sets below the observer's western horizon. These comments are applicable to all celestial objects for which $\delta < (90° - \Phi)$. Stars whose declinations meet this condition transit the observer's meridian at intervals of exactly one sidereal day; but the intervals between the successive transits of planets, the Sun and the Moon vary, although predictably.

Stars for which $\delta \geqslant (90° - \Phi)$ constitute the observer's circumpolar star field which, as viewed from *O* or any point on the Earth, rotates counterclockwise for a northern hemisphere observer or clockwise for a southern hemisphere observer as time progresses. Such stars neither rise nor set, being always above the observer's horizon as they move along small circles centered on the celestial pole. Their zenith distances have a maximum value of $180° - (\Phi + \delta)$ at lower culmination and a minimum value $(\Phi - \delta)$ or $(\delta - \Phi)$ at upper culmination as they transit the observer's meridian twice per sidereal day.

Usually, a practitioner of celestial navigation employs a marine sextant or an aircraft bubble sextant to determine the instantaneous altitude of a celestial object, the vertical angle of the object above the local horizontal plane. After appropriate corrections,[1,2] this quantity is denoted by *h*. However, because of its simpler geometrical significance, I prefer the zenith distance ζ to represent the observed quantity, with

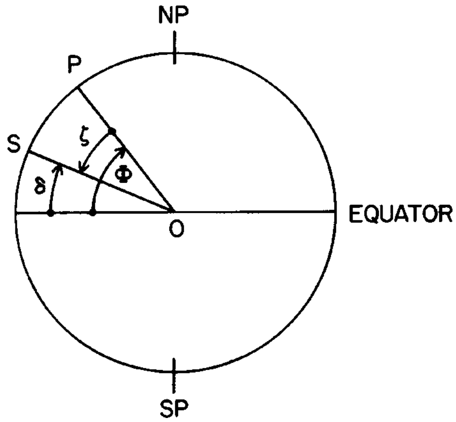

Fig. 4. P represents the position of the observer and S the position of the substellar point of a star at the moment that it transits the observer's meridian. O is the center of the Earth, δ the declination of the star, Φ the latitude of the observer, and ζ the zenith distance of the star. NP is the north pole and SP is the south pole.

$$\zeta = 90° - h. \tag{2}$$

The quantity ζ is treated as intrinsically positive and is in the range 0° to 90°.

V. DETERMINATION OF LATITUDE BY MERIDIAN TRANSIT OF A STAR

As illustrated in Fig. 4,

$$\Phi = \delta + \zeta_{\min}, \tag{3}$$

where Φ is the observer's latitude, δ is the declination of the star, and $\zeta_{\min}$ is the observed minimum value of the zenith distance of the star (a southerly angle in this example) as it transits the observer's meridian.

If the star transits the observer's meridian northward of P, the corresponding relation is

$$\Phi = \delta - \zeta_{\min}. \tag{4}$$

For a celestial object having $\delta = \Phi$, the value of $\zeta_{\min}$ is zero at the moment that the object transits the observer's meridian, that is, at the observer's zenith. Thus, in this special case, the observer's latitude is equal to the declination of the star.

The same considerations are applicable to determining latitude by observing the meridian transit of planets or the center point of the disk of the Sun. But stars are conceptually simpler because their values of α and δ are almost constant, whereas these coordinates for all celestial objects of the solar system have changing values of α and δ which must be taken from a daily ephemeris.[9]

In any case, an observer also obtains the "true north" or "true south" direction, namely the direction of any celestial object at the moment of its upper or lower culmination.

VI. DETERMINATION OF LATITUDE BY OBSERVING POLARIS

An especially valuable star for northern hemisphere navigators is the bright star Polaris (the North Star or α Ursa Minoris) which presently has declination $\delta = +89.27°$. Thus, the observer's latitude Φ is approximately equal to the altitude of Polaris or

$$\Phi \approx 90° - \zeta, \tag{5}$$

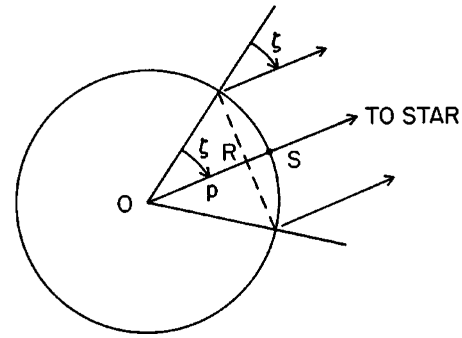

Fig. 5. Cross-section of the terrestrial sphere in any plane containing the line from the center of the Earth O through a substellar point S. The circle of position is the intersection with the sphere of the circular cone whose axis is OS and whose half-angle is ζ, the zenith distance of the star. The dashed line represents an edge-on view of the plane containing the circle of position. The normal to this plane through O has length $OR = p$.

where ζ is the zenith distance of Polaris, and, at this level of approximation, is constant and independent of the time of day. Hence, Polaris has had a special simplicity for determining latitude throughout maritime history. For example, a vessel can sail from Europe to America without an accurate knowledge of the time by simply maintaining a westerly course during which ζ of Polaris is approximately constant or changes day-by-day in such a way as to assure landfall in America at approximately an intended latitude. Because the diurnal motion of Polaris lies along a small circle around the north celestial pole of angular radius 0.73°, an improvement in accuracy in determining the latitude is to observe Polaris at specific moments identified from its ephemeris.[10,11] At lower culminations $\Phi = 90.73° - \zeta$, at upper culminations $\Phi = 89.27° - \zeta$, and at maximum eastern or western elongations, $\Phi = 90° - \zeta$. Also the observation of Polaris provides a convenient method for determining true north. Unfortunately, there is no comparably bright star in the near vicinity of the south celestial pole.[2]

Note that on the terrestrial sphere, one arcminute ($1'$) of latitude Φ corresponds to a north–south distance of one nautical mile or 6080 ft (easily remembered as "a mile a minute"), whereas one arcminute of longitude Λ corresponds to $\cos\Phi$ nautical mile.

VII. DETERMINATION OF LONGITUDE AND LATITUDE BY OBSERVATION OF TWO OR MORE STARS

Suppose that the zenith distance ζ_1 of a star has been observed at a particular time. It is evident from Fig. 5 that the same value of ζ would have been observed simultaneously at an infinite number of other points on a small circle, called the "circle of position," on the terrestrial sphere. This circle is the intersection with the sphere of a circular cone having its vertex at the center of the Earth, half angle ζ, and its axis through the substellar point S, the point on the terrestrial sphere at which the line from O to the star pierces that sphere.

Next, suppose that the zenith distance ζ_2 of a second star is observed simultaneously. This observation defines a second small circle on the terrestrial sphere. The plane containing the circle of position for star 1 and the one containing the circle of position for star 2 intersect in a line through P and P' (see Fig. 6). The observer's latitude is that of either point

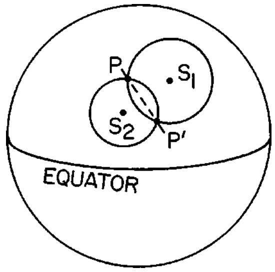

Fig. 6. In this sketch of the terrestrial sphere, $S1$ and $S2$ are the simultaneous geographic positions of the respective substellar points of star 1 and star 2. The two circles of position correspond to the observed zenith distances of the two stars and intersect at points P and P', the two possible geographic positions of the observer.

P or point P'. Auxiliary information can make possible the choice between usually widely separated points P and P'. If not, simultaneous observation of the zenith distance ζ_3 of a third star resolves the twofold ambiguity and provides a unique choice between P and P'.

The resulting position is referenced to the substellar points of the stars at the moment of observation. But these points move progressively westward as the celestial sphere rotates and the derived position of the observer also moves progressively westward at the known, constant latitude, that is, along a small circle on the terrestrial sphere in a plane parallel to the equator.

To determine the longitude Λ of the observer, a knowledge of the simultaneous longitudes of the respective substellar points is essential. In other words, it is essential to know the *absolute time* (for example, GMT, the mean solar time at Greenwich). The historical challenge of developing an accurate chronometer for maritime use rests on this simple fact.[12]

Truly simultaneous observations of stars may not be feasible in practice. But we make this assumption so as to present the basic geometric principle. Suppose that the observer has made simultaneous observations of the zenith distances of two (or more) stars at a particular moment, as described above, and that the GMT of the moment of observation also has been recorded.

It is usual to specify the hour angle of the "mean Sun" relative to the prime meridian as GMT or Universal Time (UT). The mean Sun is a fictitious point on the celestial equator that moves eastward on the celestial sphere at a constant rate so as to match the yearly average, or mean, rate of the actual Sun.

The definition of a Greenwich hour angle (GHA) is the dihedral angle measured westward from the prime meridian to any other specified meridian. The Greenwich Sidereal Time (GSiT) is the instantaneous Greenwich hour angle of the vernal equinox, GHAγ, which increases at the rate of 360° in 23 hour 56 minute 4 second (the sidereal rotational period of the Earth) or 15.04108° per mean solar hour. The values of the GHA and δ for the real Sun, the Moon, and selected planets are tabulated at 10 min intervals for each day of the year together with interpolation tables in Ref. 9.

The Greenwich Hour Angle of a celestial object, GHA*, is given by

$$\text{GHA}^* = \text{GHA}\gamma + \text{SHA}^*. \tag{6}$$

The corresponding substellar point on the terrestrial sphere is located at latitude δ and east longitude λ, where

$$\lambda = 360° - \text{GHA}^*. \tag{7}$$

The observed value of ζ plus the GHA* for each observed star complete the basic data set required for determination of the latitude and longitude of the observer. (The word "star" is used for convenience in this and other sections, but the analysis is equally applicable to any celestial body, (except the Moon, as noted above) given its instantaneous values of δ and GHA).

The following solution of the two-star-sight problem is adapted from Ref. 13. Figure 6 depicts two intersecting circles of position on the terrestrial sphere. A unique position of the observer occurs in the special case of either internal or external tangency of the two circles. But, in general the circles intersect at two points P and P'. Each circle is the intersection of a circular cone of half angle ζ with the terrestrial sphere of radius unity, or otherwise stated, the intersection of the sphere with a plane perpendicular to the line $\overline{OS}$ from the center of the sphere O through the substellar point S and at a normal distance p from O (see Fig. 5). In Fig. 5 it is noted that a line from any point on the terrestrial sphere to the identified star on the celestial sphere is parallel to $\overline{OS}$ (that is, zero parallax).

The equation of the plane containing a circle of position is[6]

$$x \cos l + y \cos m + z \cos n - p = 0, \tag{8}$$

where l, m, and n are the angles to the geographic X, Y, and Z axes. respectively. of the normal to the plane through O; p is the length of the normal $\overrightarrow{OR}$ (see Fig. 5). Also

$$p = \cos\zeta, \tag{9}$$

and

$$n = 90 - \delta. \tag{10}$$

Let

$$a \equiv \cos l = \cos\lambda \cos\delta, \tag{11a}$$

$$b \equiv \cos m = \sin\lambda \cos\delta, \tag{11b}$$

$$c \equiv \cos n = \sin\delta, \tag{11c}$$

$$p = \cos\zeta. \tag{11d}$$

In terms of the known quantities ζ, δ, and λ, the equation of each plane is

$$a_i x + b_i y + c_i z - p_i = 0, \tag{12}$$

with $i=1$ for one of the planes and $i=2$ for the other. The two planes intersect along a line PP', given by the simultaneous solution of Eq. (12) for $i=1, 2$. The result is

$$x = \frac{-Bz + C}{A} = \frac{-Ey + F}{D}, \tag{13}$$

where

$$A \equiv (a_1 b_2 - a_2 b_1), \tag{14a}$$

$$B \equiv (b_2 c_1 - b_1 c_2), \tag{14b}$$

$$C \equiv (b_2 p_1 - b_1 p_2), \tag{14c}$$

$$D \equiv (a_1 c_2 - a_2 c_1), \tag{14d}$$

$$E \equiv (b_1 c_2 - b_2 c_1), \tag{14e}$$

$$F \equiv (c_2 p_1 - c_1 p_2). \tag{14f}$$

In Eq. (13), x may be regarded as the parameter that designates a general point on the line $\bar{P}P'$ and y and z are the other two coordinates of the point. For any x, we have

$$y=\frac{F-Dx}{E}, \tag{15a}$$

$$z=\frac{C-Ax}{B}. \tag{15b}$$

The two possible positions of the observer P and P' are the intersections of the above line with the unit sphere; that is, they are the two points for which

$$x^2+y^2+z^2=1. \tag{16}$$

The substitution of Eq. (15) into Eq. (16) yields a quadratic equation for x with two real roots x_p and $x_{p'}$. The corresponding values of y_p and z_p and $y_{p'}$ and $z_{p'}$ are calculated by Eq. (15) to define the two possible positions $P\ (x_p, y_p, z_p)$ and $P'\ (x_{p'}, y_{p'}, z_{p'})$.

Finally, the latitude Φ and longitude Λ, of P and P' are calculated from the Cartesian coordinates x, y, and z by the relations

$$\tan\Phi=\frac{z}{\sqrt{x^2+y^2}}, \tag{17a}$$

$$\tan\Lambda=\frac{y}{x}, \tag{17b}$$

with attention to resolving quadrant ambiguity, to yield the desired results: $\mathrm{P}(\Phi_p, \Lambda_p)$ and $P'\ (\Phi_{p'}, \Lambda_{p'})$.

Recall that the input data for the above analytical solution are as follows. (a) From the observer: ζ_1 and ζ_2, the simultaneously observed zenith distances of the two stars and the observer's Greenwich Mean Time of the stellar observations. (b) Derived from Ref. 9: the values of the longitude λ_1 and latitude δ_1 for the substellar point of star 1 and the longitude λ_2 and the latitude δ_2 for the substellar point of star 2, both at the GMT of the observations.

The above analytical solution is too tedious to be performed by hand by a practicing navigator, but it can be easily programmed for a computer of modest capability. The time-independent values of the latitudes δ_1 and δ_2 of the two substellar points are taken from standard tables[9,10] and the time-dependent values of their longitudes λ_1 and λ_2 are given by

$$\lambda=360°-\mathrm{GHA}^* \tag{18a}$$

or

$$\lambda=360°-(\mathrm{SHA}^*+\mathrm{GHA}\gamma). \tag{18b}$$

In Eq. (18b) SHA* and GHAγ are found (with interpolation if necessary) from Ref. 9 for the observer's recorded GMT (Fig. 7).

The observer then enters the values of ζ_1, δ_1, and λ_1 and ζ_2, δ_2, and λ_2, into the computer, which yields Λ_p, Φ_p, $\Lambda_{p'}$, and $\Phi_{p'}$. If it is not obvious whether P or P' is the desired solution, the process is repeated for another pair of stars, say, 1 and 3. Note that the method is the same for any practical combination of stars, planets, and the Sun.

The traditional practice[1,2] of celestial navigation uses observations to refine the observer's position relative to an assumed position, based on dead reckoning or other approximate information. The distinctive feature of the present solution is that it obviates the need for such prior knowledge.

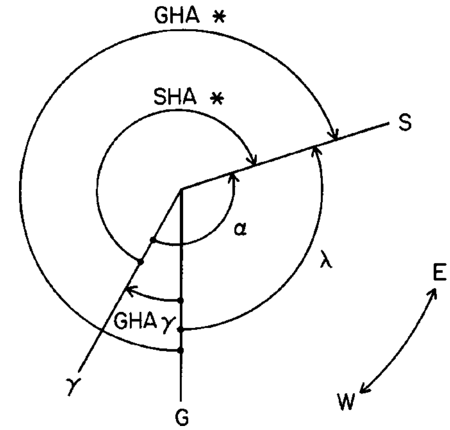

Fig. 7. GHAγ, SHA*, GHA*, α and λ are all dihedral angles (Ref. 7) as defined in Secs. II and III. Their relationships are most simply represented by this diagram of the equatorial plane in which γ is the vernal equinox, G is the intersection of the Greenwich meridian with the equator, and S is the intersection of the meridian through the star with the equator.

VIII. NOTES ON PRACTICALITY

The use of a marine sextant requires simultaneous visibility of the celestial object and the local horizon, that is, at least partially clear skies and twilight conditions for stars and planets or daylight conditions for the Sun. Under favorable conditions at sea a skilled observer can determine ζ to an accuracy of about one arcminute or better. A bubble aircraft sextant provides an artificial horizon and greatly expands the opportunities for observation, although usually with lesser accuracy than that provided by a marine sextant. Observation, when practical, of a given celestial object at intervals of, say, a few hours is equivalent to the two-star-sight technique, although movement of the observer between observations must be taken into account.

It should be mentioned that the simultaneous observation of the zenith distance and azimuth of a single celestial object at a given GMT, as is possible with a pier-mounted theodolite at a fixed point, yields both Λ and Φ. The solution of the corresponding geometric problem is straightforward, but is not within the scope of this paper. Note that an error in GMT of four seconds of time is equivalent to an error of one arcminute of longitude.

IX. SOME EXAMPLES

A. Φ by meridian transit

(1) In a sequence of twilight observations, an observer finds that the minimum value of the northerly zenith distance (that is, at meridian transit) of the bright star Vega, $\zeta_{\min}=25.51°$. From tables, the declination of Vega $\delta=+38.38°$. Hence, the observer's latitude is given by Eq. (4), namely,

$$\Phi=38.38°-25.51°=+12.87°. \tag{19}$$

(2) In a sequence of twilight observations, an observer finds the minimum zenith distance of Polaris (at upper culmination) to be $\zeta_{\min}=42.15°$. The declination of Polaris is $\delta=89.27°$. Hence,

Table I. Calculated values of zenith distances for an adopted position and GMT. Greenwich Date: 19 February 2004. Greenwich Mean Time: 20:00. GHAγ=89.11°. [All numerical data are in degrees (°).]

	SHA	δ	GHA*	λ	ζ
Sirius	258.67	−16.72	347.78	12.22	70.45
Procyon	245.12	+5.22	334.23	25.77	61.50
Aldebaran	290.95	+16.52	20.06	339.94	26.87
Pollux	243.60	+28.02	332.71	27.29	48.02

$$\Phi = 89.27° - 42.15° = +47.12°. \tag{20}$$

(3) The minimum zenith distance of the center of the Sun's disk (that is, at local noon) is observed on 21 June to be 38.35° to the north of the zenith. On that date the Sun's declination is +23.44°. Hence, $\Phi = 23.44° - 38.35° = -14.91°$ (that is, south latitude).

B. Latitude from simultaneous observation of the zenith distances ζ_1 and ζ_2 of two stars

The essential data for determining an observer's latitude Φ are the observed values ζ_1 and ζ_2 and the time-independent values of δ and SHA for the two stars (see Sec. VII).

C. Determination of both latitude Φ and longitude Λ

During evening twilight on 19 February 2004, the navigator on an imaginary ship in the central Atlantic Ocean observed simultaneously the zenith distances of four bright stars—Sirius, Procyon, Aldebaran, and Pollux—all at exactly 20:00 GMT. A copy of the Air Almanac[9] for 2004 is available. The job of the navigator is to report the ship's latitude and longitude to the captain of the ship.

In lieu of an actual set of zenith distances, I have adopted a position of the ship (unknown to the navigator) and have calculated the four relevant zenith distances. These values are then treated as "observed" values. The navigator then fills out Table I. The right-hand column of Table I lists the respective values of zenith distance ζ, calculated by the author, but regarded as "observed" values. The values of GHAγ, SHA*, and δ are taken directly from Ref. 9. The values of GHA* are found by Eq. (6) and the values of λ by Eq. (7). See also Fig. 7. Recall that δ and λ are, respectively, the constant latitude and the time-dependent longitude of the substellar point of each star at the moment of observation.

The navigator then uses the suggested computer program to calculate the latitude Φ and longitude Λ of the two possible positions of the ship for each of the six pairs of stars. For each pair there are six numbers to enter into the computer, namely ζ_1, δ_1, and λ_1 for star 1 and ζ_2, δ_2, and λ_2 for star 2, etc. The output of the computer program is given in Table II.

In every two-star case, there is the twofold ambiguity we discussed in Sec. VII. But it is clear that only a gross knowledge of the ship's position (for example, in the North Atlantic Ocean or on a mountain in Tibet) would make it possible to select the appropriate one of the two mathematically possible solutions. The uniqueness of such a choice becomes overwhelmingly clear as the other five cases are examined.

Table II. The geographic coordinates of the observer as derived from the data of Table I. [All numerical data are in degrees (°).]

		First solution	Second solution
Sirius—Procyon	Λ	−30.00	+85.16
	Φ	+42.00	−11.99
Sirius—Aldebaran	Λ	−30.00	−47.99
	Φ	+42.00	+21.84
Sirius—Pollux	Λ	−30.00	+76.65
	Φ	+42.00	+13.53
Procyon—Aldebaran	Λ	−34.79	−30.00
	Φ	−6.06	+42.00
Procyon—Pollux	Λ	−30.00	+83.96
	Φ	+42.00	+36.23
Aldebaran—Pollux	Λ	−6.82	−30.00
	Φ	−6.94	+42.00

X. CONCLUDING REMARKS

The analytical solution of the two-star sight problem requires only (corrected) observed values of the zenith distances at recorded GMT and basic astronomical data from the annual *Air Almanac*.[9] And, in practice, it requires a computer of modest capability. A single program covers all cases. However, it does not require an assumed position, massive sight reduction tables, or massive tables of computed altitude and azimuth.

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APPENDIX: SUGGESTED EXERCISES FOR THE STUDENT

(1) Verify the astronomical data in the first four columns of Table I. Then use the author's adopted position $\Phi = +42.00°$, $\Lambda = -39.00°$ to calculate the four zenith distances in the fifth column.
(2) Write a computer program for solving the two-star-sight problem. Then use the data in Table I and verify the author's calculated values of Φ and Λ in Table II.
(3) Show, for a given date of the year, that the time interval between sunrise and sunset yields a crude measure of an observer's latitude Φ. Estimate the accuracy by your own observations.

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[1]*American Practical Navigator*, H.O. Publ. No. 9, U.S. Navy Hydrographic Office (USGPO, Washington, DC, 1966).
[2]F. W. Wright, *Celestial Navigation* (Cornell Maritime Press, Cambridge, MD, 1976).
[3]R. M. Green, *Spherical Astronomy* (Cambridge U.P., Cambridge, 1985).

[4]This approximation ignores the oblateness of the actual Earth as well as the heights of mountains, etc., and thereby eliminates the distinctions among astronomical latitude, geocentric latitude, and geodetic latitude.

[5]D. T. Sigley and W. T. Stratton, *Solid Geometry* (Dryden, New York, 1956), revised edition.
[6]H. G. Ayre and R. Stephens, *A First Course in Analytic Geometry* (Van Nostrand, Princeton, NJ, 1956).
[7]A dihedral angle is the ‘‘hinge’’ angle between two intersecting planes. It is measured by the plane angle between two lines that pass through a common point on the line of intersection of the planes, one line lying in each plane and perpendicular to the line of intersection of the two planes.
[8]A primitive technique for determining longitude without the use of an accurate time piece exploits this parallactic effect (‘‘linear distance’’ technique).
[9]*The Air Almanac 2004*, The National Almanac Office, United States Naval Observatory (USGPO, Philadelphia, PA, 2003).
[10]*The Astronomical Almanac for the Year 2004*, The Nautical Almanac Office, United States Naval Observatory (USGPO, Washington, DC, 2002).
[11]*Observer's Handbook 2004* (The Royal Astonomical Society of Canada, Toronto, 2003).
[12]D. Sobel, *Longitude* (Walker, New York, 1995).
[13]J. A. Van Allen, ‘‘An analytical solution of the two star sight problem of celestial navigation,’’ Navig.: J. Inst. Navig. **28**(1), 40–43 (1981).

Horizontal Electromagnet. This horizontal helix on a stand was probably made by Daniel Davis or one of his successors at some time after ca. 1850. The hollow coil produces a magnetic field that is enhanced when an iron bar is placed inside. The 1842 edition of Daniel Davis's *Manual of Magnetism* notes that ‘‘the magnetizing power will be greatly increased if the wire be coiled in the manner of a cork-screw, so as to form a hollow cylinder into which the body to be magnetized can be inserted. Such a coil is denominated a Helix.’’ The helix is in the Greenslade Collection. (Photograph and notes by Thomas B. Greenslade, Jr., Kenyon College)